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Some Remarks on Total Categories

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INTRODUCTION

Let *set* denote the category of small sets. A category \mathcal{B} is said to have *small hom sets* if it has a hom functor $(-, -): \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow \text{set}$. For such \mathcal{B} we will write

$$Y: \mathcal{B} \rightarrow \hat{\mathcal{B}} = \text{set}^{\mathcal{B}^{\text{op}}} \quad \text{and} \quad Z: \mathcal{B} \rightarrow \check{\mathcal{B}} = (\text{set}^{\mathcal{B}})^{\text{op}}$$

for the associated Yoneda embeddings. (In general, $\hat{\mathcal{B}}$ and $\check{\mathcal{B}}$ do not have small hom sets.) Following Street and Walters [3], a category \mathcal{B} is said to be *total* (respectively *cototal*) if \mathcal{B} has small hom sets and Y (respectively Z) has a left (respectively right) adjoint. We will reserve L (respectively R) for such adjoints. Of course \mathcal{B} is cototal if and only if \mathcal{B}^{op} is total.

Totality is a strong form of cocompleteness. In the interests of completeness (in some sense or other) we recall a few basic results from [3, 6]. For $\Phi \in \hat{\mathcal{B}}$, $\Phi L \simeq \varinjlim (\Phi \text{el} \rightarrow \mathcal{B})$, where Φel denotes the usual category of elements of Φ . Any functor $D: \mathcal{C} \rightarrow \mathcal{B}$ admits a factorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{E} & \mathcal{E} \\ & \searrow D & \downarrow M \\ & & \mathcal{B} \end{array}$$

with E final and M a discrete fibration (see [4] or [2]). It follows that for \mathcal{B} with small hom sets: \mathcal{B} is total if and only if every diagram in \mathcal{B} , whose associated discrete fibration has small fibres, has a colimit. Furthermore, for any \mathcal{B} , $\hat{\mathcal{B}}$ is small bicomplete. So any total category, being a full reflective subcategory of a small bicomplete category, is small bicomplete. A category with small hom sets which is small bicomplete need not be total though. (A counterexample which shows somewhat more is given in Section 1.)

An indication of the strength of totality is provided by Theorem 1 below. Recall that a functor $F: \mathcal{B} \rightarrow \mathcal{X}$ is *admissible* if $\mathcal{X}(BF, X)$ is small for all $B \in \mathcal{B}$ and $X \in \mathcal{X}$. If \mathcal{X} has small hom sets any such F is admissible.

THEOREM 1 (Street and Walters). *If \mathcal{B} is total and $F: \mathcal{B} \rightarrow \mathcal{X}$ is admissible, then F has a right adjoint if and only if F preserves all colimits.*

A wealth of examples is provided by the next result.

THEOREM 2 (Street and Walters). (i) *set is total.*

(ii) *If \mathcal{B} is total and \mathcal{A} is small, then $\mathcal{B}^{\mathcal{A}}$ is total.*

(iii) *If \mathcal{C} is a full reflective subcategory of \mathcal{B} and \mathcal{B} is total, then \mathcal{C} is total.*

Totality is a notion which makes sense in a large class of 2-categories. Indeed, it was at an abstract level that the definition was given in [3]. "Total object relative to a Yoneda structure" promises to unify many cocompleteness notions. At this time, however, it seems reasonable to pursue the concept for categories. Our remarks in Section 4 can be construed as attempts to identify the total objects in some closely related 2-categories.

1. DUALITY

For a small category \mathcal{A} one has the Isbell conjugation functors

$$\begin{array}{ccc} & ()^+ & \\ \mathcal{A} & \xrightarrow{\quad} & \mathcal{A}^{\mathcal{A}} \\ & \perp & \\ & \xleftarrow{\quad} & \\ & ()^- & \end{array}$$

defined by $A\Phi^+ = \text{set}^{\mathcal{A}^{\text{op}}}(\Phi, (-, A))$ and $A\Psi^- = \text{set}^{\mathcal{A}}(\Psi, (A, -))$ for $\Phi \in \hat{\mathcal{A}}$, $\Psi \in \check{\mathcal{A}}$ and $A \in \mathcal{A}$. Clearly $()^+$ and $()^-$ commute with Y and Z . For a general \mathcal{B} they do not exist, but if \mathcal{B} is total we can define $()^+$ by the above formula, since in that case $\hat{\mathcal{B}}(\Phi, (-, B)) \simeq \mathcal{B}(\Phi L, B)$ is small. Similarly, if \mathcal{B} is cototal $()^-$ exists. In conjunction with the adjointness relation $\hat{\mathcal{B}}(\Phi^+, \Psi) \simeq \hat{\mathcal{B}}(\Phi, \Psi^-)$, when $()^+$ and $()^-$ exist, the following formulas are useful:

PROPOSITION 3. *For $\Phi \in \hat{\mathcal{B}}$, $\Psi \in \check{\mathcal{B}}$.*

(i) *If \mathcal{B} is total, $\hat{\mathcal{B}}(\Psi, \Phi^+) \simeq (\Phi L)\Psi$.*

(ii) *If \mathcal{B} is cototal, $\hat{\mathcal{B}}(\Psi^-, \Phi) \simeq (\Psi R)\Phi$.*

Proof. (i) $\check{\mathcal{B}}(\Psi, \Phi^+) \simeq \int_X \text{set}(X\Phi^+, X\Psi) \simeq \int_X \text{set}(\hat{\mathcal{B}}(\Phi(-, X), X\Psi) \simeq \int_X \text{set}((\Phi L, X), X\Psi) \simeq (\Phi L)\Psi$.

(ii) Similarly.

PROPOSITION 4. For $\Phi \in \hat{\mathcal{B}}$, $\Psi \in \check{\mathcal{B}}$. If \mathcal{B} is total and cototal, then

$$\begin{array}{ccc} \check{\mathcal{B}}(\Phi^+, \Psi) & \simeq & \hat{\mathcal{B}}(\Phi, \Psi^-) \\ \parallel & & \parallel \\ (\Phi L) \Psi^- & & (\Psi R) \Phi^+ \\ \searrow & & \swarrow \\ & \mathcal{B}(\Phi L, \Psi R). \end{array}$$

Proof. $\hat{\mathcal{B}}(\Phi, \Psi^-) \simeq \int_X \text{set}(X\Phi, X\Psi^-) \simeq \int_X \text{set}(X\Phi, \check{\mathcal{B}}((X, -), \Psi)) \simeq \int_X \text{set}(X\Phi, (X, \Psi R)) \simeq \hat{\mathcal{B}}(\Phi, (-, \Psi R)) \simeq (\Psi R) \Phi^+ \simeq \mathcal{B}(\Phi L, \Psi R)$.

If \mathcal{B} is an ordered set, that is, if $(-, -): \mathcal{B}^{\text{op}} \times \mathcal{B} \rightarrow 2$, it makes sense to adjust the Yoneda arrows accordingly. Y (respectively Z) becomes the embedding of \mathcal{B} in the set of left (respectively right) order ideals of \mathcal{B} . Now \mathcal{B} is sup complete if and only if Y has a left adjoint. For a right ideal J , J^- is the left ideal of lower bounds of J . A familiar calculation shows that $JR \simeq J^-L$. In other words, if \mathcal{B} is sup complete, it is also inf complete. For general \mathcal{B} the analogue of this statement is false, but if \mathcal{B} is total and cototal the formula above is correct.

THEOREM 5. If \mathcal{B} is total and cototal,

(i) $R \simeq (-)^- L$ and

(ii) $L \simeq (-)^+ R$.

Proof of (i). For $\Psi \in \check{\mathcal{B}}$, $B \in \mathcal{B}$. $\mathcal{B}(\Psi R, B) \simeq \hat{\mathcal{B}}((- , \Psi R), (- , B)) \simeq \int_X \text{set}((X, \Psi R), (X, B)) \simeq \int_X \text{set}(\check{\mathcal{B}}((X, -), \Psi), (X, B)) \simeq \int_X \text{set}(X\Psi^-, (X, B)) \simeq \hat{\mathcal{B}}(\Psi^-, (- , B)) \simeq \mathcal{B}(\Psi^-L, B)$.

The above proof also shows that $(-)^{-+} \simeq RZ$. Z , being fully faithful, is cotripleable, so in the context of bitotality L is the canonical comparison functor.

The following “natural” example of a total category which is not cototal was shown to the author by Bob Paré: The category of small groups, *grp*, is easily seen to be total by Theorem 2. For each infinite cardinal α let S_α denote a simple group of cardinality α and consider the diagram suggested by

$$\begin{array}{ccc} & (S_\alpha, -) & \\ (1, -) & \nearrow & \cdot \\ & & \cdot \\ (1, -) & & \cdot \end{array}$$

in set^{grp} . Evaluating the diagram at an arbitrary group G produces a diagram of sets almost all of which are 1. Thus the diagram has a limit in grp . Were R to exist for this example so also would $\prod_a S_a$ in grp , which is not the case since the forgetful functor $grp \rightarrow set$ preserves all limits.

2. EXISTENCE

The next result provides a reasonable number of examples of bitotal categories. Further examples of such will be given in Section 3.

THEOREM 6. *If \mathcal{B} is total and has a small set of cogenerators, then \mathcal{B} is cototal.*

Proof. Z preserves all colimits so, by Theorem 1, we have only to show that it is admissible.

Let \mathcal{C} denote the small full subcategory of \mathcal{B} determined by a small set of cogenerators. Then for all $X \in \mathcal{B}$ the canonical morphism

$$X \rightarrow \int_C \{(X, C), C\},$$

where the integral is over $C \in \mathcal{C}$ and $\{, \}$'s denote powers, is mono. For all $B \in \mathcal{B}$ and $\Psi \in \mathcal{B}$ we have

$$\begin{aligned} \mathcal{B}(BZ, \Psi) &\simeq set^{\mathcal{B}}(\Psi, (B, -)) \twoheadrightarrow set^{\mathcal{B}}(\Psi, (B, \int_C \{(-, C), C\})) \\ &\simeq \int_C set^{\mathcal{B}}(\Psi, ((-, C), (B, C))) \simeq \int_C \mathcal{B}((- , C), (\Psi, (B, C))) \\ &\simeq \int_C (C\Psi, (B, C)) \in set. \end{aligned}$$

Any Grothendieck topos has a small set of cogenerators (if $(G_i)_{i \in I}$ generate, $(\Omega^{G_i})_{i \in I}$ cogenerate), so Grothendieck topoi are bitotal categories. The next result is, in a sense, similar to Theorem 6.

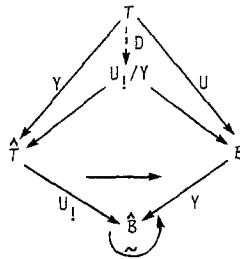
THEOREM 7. *If \mathcal{A} is small, $D: \mathcal{A} \rightarrow \mathcal{B}$ is dense and \mathcal{B} is small cocomplete, then \mathcal{B} is total.*

Proof. D being dense, $(D, -): \mathcal{B} \rightarrow \mathcal{A}: B \mapsto \mathcal{B}(-D, B)$ is fully faithful. It is a simple matter to check that $\Gamma \mapsto \int^A A\Gamma \cdot AD$, where the integral is over $A \in \mathcal{A}$ and $- \cdot -$'s denote multiples, is left adjoint to it.

3. TOPOLOGICAL EXAMPLES

It is instructive to begin by reviewing the construction of colimits in *top*, the category of small topological spaces, and related examples. To form $\varinjlim M$ for $M: \mathcal{E} \rightarrow \text{top}$, one composes with $U: \text{top} \rightarrow \text{set}$, the underlying set functor; forms the colimit $\iota: MU \rightarrow \varinjlim MU$ in *set*; and finally equips $\varinjlim MU$ with the finest topology for which all components of ι are continuous. The definition which follows abstracts this last mentioned lifting of ι .

A functor $U: \mathcal{E} \rightarrow \mathcal{B}$ is said to be a *total opfibration* if \mathcal{E} has small hom sets; \hat{U} has a left adjoint, $U_!$; and U satisfies the following *lifting condition*: The functor $D: \mathcal{E} \rightarrow U_!/Y$ defined by



has a left adjoint over \mathcal{B} .

To understand the condition recall first that an object $\Phi \in \mathcal{E}$ is coextensive with a diagram $M: \mathcal{E} \rightarrow \mathcal{E}$ in \mathcal{E} , where M is a discrete fibration with small fibres. Then $\Phi U_!$ corresponds to the diagram obtained by factoring MU , and an object $(\Phi, \gamma: \Phi U_! \rightarrow (-, B), B)$ of $U_!/Y$ corresponds to a cone from this diagram to B . The hypothesized left adjoint for D , together with its unit, provides, for such an object of $U_!/Y$, an object $T \in \mathcal{E}$ and a “cone” $\Phi \rightarrow (-, T)$ which is a best lifting of γ . The paradigmatic example of a total opfibration is of course $U: \text{top} \rightarrow \text{set}$.

THEOREM 8. *If $U: \mathcal{E} \rightarrow \mathcal{B}$ is a total opfibration and \mathcal{B} is total, then \mathcal{E} is total and U is cocontinuous.*

Proof. Each of the factors of Y in the diagram above has a left adjoint. That of $P: U_!/Y \rightarrow \mathcal{E}$ is obtained by general comma nonsense from L , the left adjoint of \mathcal{B} 's Y . Explicitly it sends $\Phi \in \mathcal{E}$ to $(\Phi, \eta: \Phi U_! \rightarrow (-, \Phi U_! L), \Phi U_! L) \in U_!/Y$. This description, together with the fact that the left adjoint of D is over \mathcal{B} , shows that U is cocontinuous.

It is clear that the two step construction of L for \mathcal{E} is only a generalization of the familiar procedure recalled at the beginning of this section.

A somewhat weaker definition of total opfibration could have been given. Recall from [3] that a functor $U: \mathcal{E} \rightarrow \mathcal{B}$ is *total* if it is admissible and $(U, -)$ (which thus exists) has an admissible left adjoint, which we will

denote by $- \cdot U$. For total U and \mathcal{E} with small hom sets, the lifting condition above can be expressed by requiring that D , defined by

$$\begin{array}{ccc}
 & T & \\
 \gamma \swarrow & \downarrow D & \searrow U \\
 T & - \cdot U / \mathcal{B} & \mathcal{B} \\
 \downarrow - \cdot U & \xrightarrow{\quad} & \downarrow \mathcal{B} \\
 & \mathcal{B} & \\
 & \downarrow U & \\
 & \mathcal{B} &
 \end{array}$$

where \tilde{U} corresponds, via

$$\frac{Y \cdot U \rightarrow U}{Y \rightarrow (U, U)},$$

to the strength of U , has a left adjoint over \mathcal{B} . $- \cdot U / \mathcal{B} \rightarrow \mathcal{E}$ has a left adjoint and it follows that if \mathcal{E} has small hom sets, U is total and U satisfies the lifting condition, then \mathcal{E} is total and U is cocontinuous. From

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{Y} & \hat{\mathcal{B}} \\
 (U, -) \downarrow & \simeq & \downarrow \tilde{U} \\
 \mathcal{E} & &
 \end{array}$$

Theorem 8 now follows, but the added generality does not yet seem useful and the earlier discussion is closer to present experience.

It is a simple exercise in dualization to define a *cototal fibration* and conclude that *top*, etc., are also cototal. Note that we can now conclude that *ct2*, the category of small compact Hausdorff spaces, is bitotal. This theorem (together with other results of this paper) was presented at Oberwolfach in 1979. Subsequently, Tholen [5] showed that totality lifts through semitopological functors. Recall that $U: \mathcal{E} \rightarrow \mathcal{B}$ is semitopological if and only if every U -cone has a rigid U -semi-initial lifting. In this terminology U is a cototal fibration if and only if every U -cone, obtained from a discrete opfibration with small fibres over \mathcal{E} , has a U -initial lifting. The two notions are incomparable; each is a generalization of "topological functor."

4. CLASSIFICATION

It would seem desirable to express additional properties that a total category may have in terms of L . The following result is illustrative of what we have in mind. It is reminiscent of Day's reflection theorem [1], but note that $\hat{\mathcal{B}}$ is not cartesian closed.

THEOREM 9. *If \mathcal{B} is total, then \mathcal{B} is cartesian closed if and only if L preserves binary products.*

Proof. \mathcal{B} is cartesian closed if and only if for each $A \in \mathcal{B}$, $A \times -$ is cocontinuous. Since $(A \times -)_!$ exists and is given by $(-, A) \times -$, the latter is equivalent to $((-, A) \times \Phi)L \simeq A \times \Phi L$, for all $\Phi \in \mathcal{B}$. $(-, A)L \simeq A$, so it only remains to be shown that such isomorphisms for all $A \in \mathcal{B}$ and $\Phi \in \mathcal{B}$ imply that L preserves binary products.

$$\begin{aligned} (\Psi \times \Phi)L &\simeq \left(\left(\int^A A\Psi \cdot (-, A) \right) \times \Phi \right)L \simeq \left(\int^A A\Psi \cdot ((-, A) \times \Phi) \right)L \\ &\simeq \int^A A\Psi \cdot ((-, A) \times \Phi)L \simeq \int^A A\Psi \cdot (A \times \Phi L) \\ &\simeq \left(\int^A A\Psi \cdot A \right) \times \Phi L \simeq \Psi L \times \Phi L. \end{aligned}$$

A total category is said to be *lex total* if L preserves finite limits. Walters [6] has shown that “ \mathcal{B} is lex total and has a small set of generators” is equivalent to “ \mathcal{B} is a Grothendieck topos.”

For small \mathcal{A} , $\mathcal{A}Y: \mathcal{A} \rightarrow \mathcal{A}$ is such that $\widehat{\mathcal{A}Y}$ has both right and left adjoints. $(\mathcal{A}Y)_! \dashv \widehat{\mathcal{A}Y} \dashv \mathcal{A}Y$ expresses the fact that small powers of *set* are considerably more than lex total. The exponential transpose of $(\mathcal{A}Y)_!$, $\mathcal{A}^{\text{op}} \times \mathcal{A} \rightarrow \text{set}$ can be shown to be given by $\langle F, G \rangle \mapsto F^+ \cdot G$, $(F^+ \cdot G \simeq \int^A AG \cdot AF^+)$. That of $\mathcal{A}Y$ is $\langle F, G \rangle \mapsto \mathcal{A}(F, G)$. If idempotents split in \mathcal{A} , a familiar result says that $F \in \mathcal{A}$ is representable if and only if $F^+ \cdot F \simeq \mathcal{A}(F, F)$. This suggests a candidate for \mathcal{A} with \mathcal{A} equivalent to \mathcal{B} , if \mathcal{B} is total with $T \dashv L \dashv Y$.

Finally, we note that for $\mathcal{B} = \text{set}$ we have

$$\begin{array}{ccc} & K & \\ & \xrightarrow{\quad} & \\ & (\phi) - \perp & \\ & \xleftarrow{\quad} & \\ & \Delta \perp & \\ \text{set} & \xrightarrow{\quad} & \widehat{\text{set}} \\ & (1) - \perp & \\ & \xleftarrow{\quad} & \\ & Y \perp & \\ & \xrightarrow{\quad} & \\ & Z & \\ & \swarrow R & \\ & & ()^+ \dashv ()^- \\ & & \downarrow \\ & & \widehat{\text{set}} \end{array}$$

where $XX \simeq X \cdot (-, \phi)$ and \mathcal{A} is the usual diagonal. K does not have a left adjoint and by Theorem 9 it follows that R does not have a right adjoint. Other examples of total categories with such a configuration of adjoints do not seem to be forthcoming.

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